Appendix A. Proofs of propositions in chapter 4

***Notation*:** Since most of the proofs follow Norman (1968) we adopt his notation. The state of the system in iteration *n*, characterized in the BM model by the mixed-strategy profile in iteration *n*, is denoted S_n. The set of possible states is called the *state space* and denoted *S*. The realization of both players' decisions in iteration *n* is referred to as an event and denoted E_n. The set of possible events is called the *event space* and denoted *E*. S_n and E_n are to be considered random variables. In general, *s* and *e* denote elements of the state and event spaces, respectively. The function of *S* into *S* that maps S_n into S_{n+1} after the occurrence of event *e* is denoted $f_e(\cdot)$. Thus, if $E_n = e$ and $S_n = s$, then $S_{n+1} = f_e(s)$. Let $T_n(s)$ be the set of values that S_{n+1} takes on with positive probability when $S_1 = s$. Let us say that a state *s is associated* with an event *e* if *s* is a pure state (where all probabilities are either 0 or 1) and the occurrence of *e* pushes the system towards *s* from any other state. In any system, only one state is associated with a certain event, but the same state may be associated with several events. Finally, use $d(A, B)$ for the minimum Euclidean distance between two subsets *A* and *B* of *S*.

$$d(A,B) = \inf_{s\in A, s'\in B} d(s,s')$$

Lemma 1. Assuming players' aspiration levels are different from their respective payoffs, the 2-player 2-strategy BM model can be formulated as a strictly distance diminishing model (Norman, 1968, p.64).

Proof. Proving that the BM model can be formulated as a strictly distance diminishing model involves checking that hypotheses H1 to H8 in Norman (1968) hold. Define the state of the system S_n in iteration *n* in the BM model as the mixed-strategy profile in iteration *n*. The state space is then the mixed-strategy space of the game, and the event space *E* is the space of pure-strategy profiles, or possible outcomes of the game; consider also the Euclidean distance $d(s, s')$ in *S*. Having stated that, hypotheses H1 to H6 (which are included here for the sake of completeness) are immediate:

H1. The occurrence of an event effects a change of state such that if $E_n = e$ and $S_n = s$, then $S_{n+1} = f_e(s)$. Thus, $S_{n+1} = f_{E_n}(S_n)$ for $n \geq 1$.

H2. *E* is a finite set.

H3. The learning situation is memory-less and temporally homogeneous, in the sense that the probabilities of the various possible events on trial n depend only on the state on trial n, and not on earlier states or events, or on the trial number. That is, there is a real valued function $\varphi.(\cdot)$ on $E \times S$ such that

$$P_s(E_1 = e_1) = \varphi_{e_1}(s) \ ,$$

and $$P_s(E_{n+1} = e_{n+1} \mid E_j = e_j,\ 1 \le j \le n) = \varphi_{e_{n+1}}(f_{e_1 \ldots e_n}(s)) \quad , \text{ for } n \ge 1,$$

where $f_{e_1 \ldots e_n}(s) = f_{e_n}(f_{e_{n-1}}(\ldots(f_{e_1}(s)))$

H4. (S,d) is a metric space.

H5. (S,d) is compact.

H6. Let us use the following notations. If h and g are mappings of S into the real numbers and into S, respectively, their maximum "difference quotients" $m(h)$ and $u(g)$ are defined by

$$m(h) = \sup_{s \neq s'} \frac{|h(s) - h(s')|}{d(s,s')} \qquad \text{and} \qquad u(g) = \sup_{s \neq s'} \frac{d(g(s), g(s'))}{d(s,s')}$$

whether or not these are finite. H6 is the following regularity condition:

$$m(\varphi_e) < \infty \quad \text{for all } e \in E$$

This is easily proven by defining $\varphi_e(s) \equiv d(s,0)$

H7. For strictly distance diminishing models H7 reads

$$\sup_{s \neq s'} \frac{d(f_e(s), f_e(s'))}{d(s,s')} < 1 \qquad \text{for all } e \in E$$

Given that learning rates are strictly within 0 and 1 and stimuli are always non-zero numbers between −1 and 1 (since players' aspiration levels are different from their respective payoffs by assumption), it can easily be checked that H7 holds. The intuitive idea is that after any event e, the distance from any state s to the pure state s_e associated with event e is reduced by a fixed proportion in each of the components of s which is not already equal to the corresponding component in s_e. For the strict inequality in H7 to hold, it is instrumental that every state of the system (except at most one for each event) changes after any given event occurs (i.e. $f_e(s) \neq s$ for all $s \neq s_e$). The assumption that players' aspiration levels are different from their respective payoffs guarantees such a requirement. Without that assumption, H7 does not necessarily hold in its strict form.

Finally, H8 reads:

H8. For any $s \in S$ there is a positive integer k and there are k events e_1 ,…, e_k such that

$$\sup_{s \neq s'} \frac{d(f_{e_1 \ldots e_n}(s), f_{e_1 \ldots e_n}(s'))}{d(s, s')} < 1 \quad \text{and} \quad P(E_j = e_j, 1 \leq j \leq n \mid S_1 = s) > 0$$

where $f_{e_1 \ldots e_n}(s) = f_{e_n}(f_{e_{n-1}}(\ldots(f_{e_1}(s)))$

H8 is immediate having proved H7 in its strict form, since at least one event is possible in any state.█

Lemma 2. Consider any 2-player 2-strategy BM system where players' aspiration levels differ from all their respective payoffs. Let s_e be the state associated with event e. If e may occur when the system is in state s ($\Pr\{E_n = e \mid S_n = s\} > 0$), then

$$\lim_{n \to \infty} d(T_n(s), s_e) = 0$$

Proof. The BM model specifications guarantee that if event e may occur when the system is in state s, then it will also have a positive probability of happening in any subsequent state. Mathematically,

$$\Pr\{E_n = e \mid S_n = s\} > 0 \quad \rightarrow \quad \Pr\{E_{n+t} = e \mid S_n = s\} > 0 \qquad \text{for any } t \geq 0$$

This means that any finite sequence of events $\{e, e \ldots e\}$ has positive probability of happening. Note now that if the system is in state $s \neq s_e$ and event e occurs, the distance from s to s_e is reduced by a fixed proportion in each of the components of s which is not already equal to the corresponding component in s_e. This proportion of reduction is, for each player, the product of the player's absolute stimulus magnitude generated after e and the player's learning rate. Both proportions are strictly between 0 and 1 since players' aspiration levels are different from their respective payoffs by assumption. Let k be the minimum of those two proportions. Imagine then that event e keeps occurring, and note the following bound.

$$d(T_n(s), s_e) \leq (1-k)^n \cdot d(s, s_e)$$

The proof is completed taking limits in the expression above.

$$0 \leq \lim_{n \to \infty} d(T_n(s), s_e) \leq \lim_{n \to \infty} (1-k)^n \cdot d(s, s_e) = 0 \qquad █$$

Proof of Proposition 4-1. Statement (i) is an application of Theorem 1 in chapter 2 of Benveniste et al. (1990, p. 43). Statement (ii) follows from Theorem 8.1.1 in Norman (1972, p. 118). The assumptions to apply this Theorem are listed in Norman (1972, p. 117). Here we show that with the hypotheses in Proposition 4-1, all these assumptions hold. In this section, following Norman (1972), the state of the system in iteration n is denoted X_n, and the letter θ denotes the learning rate. Since the state space $I_\theta = I$ is independent of θ, (a.1) is satisfied. $H_n^\theta = \Delta X_n^\theta / \theta$ does not depend on θ, so (a.2) and (a.3) hold. All components of the functions $w(x) = \mathrm{E}(H_n^\theta \big| X_n^\theta = x)$ and $s(x) = \mathrm{E}((H_n^\theta - w(x))^2 \big| X_n^\theta = x)$ are polynomials, so every assumption (b) is satisfied. Finally, since H_n^θ does not depend on θ the supremum over θ can be omitted in (c), and also the module of each of the components of H_n^θ is bounded by the maximum learning rate, so (c) is also satisfied. Thus Theorem 8.1.1 is applicable. Finally, Statement (iii) is an application of Theorem 4.1 in chapter 8 of Kushner and Yin (1997).$\blacksquare$

Proof of Proposition 4-2. Proposition 4-2 follows from Theorem 2.3 in Norman (1968, p.67), which requires the model to be distance-diminishing and one extra assumption H10.

H10. There are a finite number of absorbing states a_1 ,…, a_N , such that, for any $s \in S$, there is some $a_{j(s)}$ for which

$$\lim_{n\to\infty} d(T_n(s), a_{j(s)}) = 0$$

Given the assumptions of Proposition 4-2, Lemma 1 can be used to assert that the BM model is distance diminishing, with associated stochastic processes $\boldsymbol{S}_n$ and $\boldsymbol{E}_n$. Proving that H10 prevails will then complete the proof. The proof of H10 rests on the following three points:

a) If in state s there is a positive probability of an event e occurring, then, applying Lemma 2:

$$\lim_{n\to\infty} d(T_n(s), s_e) = 0$$

where s_e is the state associated with the event e.

b) The state s_e associated with a Mutually Satisfactory (MS) event e is absorbing. Note also that there are at most four absorbing states.

c) From any state there is a positive probability of playing a MS event within three iterations.

Points (a) and (b) are straightforward. To prove (c) we define strictly mixed strategies as those that assign positive probability to both actions, and mixed states as states where both players' strategies are strictly mixed. Note that after an unsatisfactory event, every player modifies her strategy so the updated strategy is strictly mixed, and that strictly mixed strategies will always remain so.

Since players' aspiration levels are below their respective *maximin* by assumption, there is at least one MS event. Hence from any mixed state there is a positive probability for a MS event to happen. We focus then on non-mixed states where no MS event can occur in the first iteration. This implies that the event in the first iteration is unsatisfactory for at least one player, so at least one player will have a strictly mixed strategy in the second iteration. Without loss of generality let us say that player 1 has a strictly mixed strategy in the second iteration. If player 2's strategy were also strictly mixed, then the state in the second iteration would be mixed, and a MS event could occur. Imagine then that the state in the second iteration is not mixed. Given that player 1's aspiration is below its *maximin*, there is a positive probability that the event in iteration 2 will be satisfactory for player 1. If such a possible event is also satisfactory for player 2, an MS event has occurred. If not, then both players will have a strictly mixed strategy in iteration 3, so a MS event could happen in iteration 3. This finishes the proof of point (c).

The proof of the fact that every SRE can be asymptotically reached with positive probability if the initial state is completely mixed rests on two arguments: (a) there is a strictly positive probability that an infinite sequence of any given MS event *e* takes place (this can be proved using Theorem 52 in Hyslop (1965, p.94)), and (b) such an infinite run would imply convergence to the associated (SRE) state s_e. We also provide here a theoretical result to estimate with arbitrary precision the probability L_∞ that an infinite sequence of a MS event $\boldsymbol{e} = (d_1, d_2)$ begins when the system is in mixed state $\boldsymbol{p} = (p_{1,d_1}, p_{2,d_2})$.

$$L_\infty = \lim_{n\to\infty} \prod_{t=0}^{n} [1-(1-p_{1,d_1})(1-l_1 s_1(d_1))^t][1-(1-p_{2,d_2})(1-l_2 s_2(d_2))^t]$$

The following result can be used to estimate L_∞ with arbitrary precision:

Let $P_k = \prod_{t=0}^{k-1}(1-xy^t)$ and let $P_\infty = \lim_{k\to\infty} P_k$. Then, for $x, y \in (0, 1)$,

$$P_k > P_\infty > P_k\left(1-\frac{xy^k}{1-y}\right)$$

We are indebted to Professor Jörgen W. Weibull for discovering and providing the lower bound in this result (personal communication).$\blacksquare$

<u>Proof of Proposition 4-3</u>. Each statement of Proposition 4-3 will be proved separately. Statement (i) is an immediate application of Theorem 2.3 in Norman (1968, p.67), which requires the model to be distance-diminishing and the extra assumption H10 (see proof of Proposition 4-2). Having proved in Lemma 1 that the model is distance-diminishing, we prove here that H10 holds. The proof of H10 rests on the same three points (a-c) exposed in the proof of Proposition 4-2. The terminology defined there is also used here. Points (a) and (b) are straightforward. To prove (c), remember that after an unsatisfactory event, every player modifies her strategy so the updated strategy is strictly mixed, and that strictly mixed strategies always remain so. By assumption, there is at least one absorbing state, which means that there must be at least one MS event. This implies that from any mixed state there is a positive probability for a MS event to happen.

Since players' aspirations are above their respective *maximin*, given any action for player *i*, there is always an action for her opponent such that the resulting event would be unsatisfactory for player *i*. In other words, if one of the players has a strictly mixed strategy, then there is a positive chance that the system will be in a mixed state in the next iteration. We focus then on states where no player has strictly mixed strategies and a MS event cannot occur in the first iteration. This implies that the event in the first iteration is unsatisfactory for at least one player, who will have a strictly mixed strategy in the second iteration and, as just shown, this implies a positive probability that the system will be in a mixed state in the third iteration. The proof of statement (i) is then finished.

Statement (ii) follows from Theorem 2.2 in Norman (1968, p.66), which requires the model to be distance-diminishing and one extra assumption H9.

H9. $\lim_{n\to\infty} d(T_n(s), T_n(s')) = 0$ for all $s, s' \in S$

Having proved in Lemma 1 that the model is distance-diminishing, we prove here that H9 holds. Since, by assumption, there are no absorbing states, there cannot be MS events. This implies that the event in the first iteration is unsatisfactory for at least one player, who will have a strictly mixed strategy in the second iteration. As argued in the proof of statement (i), this implies a positive probability that the system will be in a mixed state in the third iteration. Therefore at the third iteration any event has a positive probability of happening, so we can select any one of them, the state s_e associated with it, and then, by Lemma 2, we know that $\lim_{n\to\infty} d(T_n(s), s_e) = 0$ for any state s, so H9 holds. █

__Proof of Proposition 4-4__. The reasoning behind this proof follows Sastry et al. (1994). Statement (i) can be proved considering one player i who benefits by deviating from the SRE by increasing her probability $p_{i,q}$ to conduct action q. The expected change in probability $p_{i,q}$ can then be shown to be strictly positive for all $p_{i,q} > 0$ while keeping the other player's strategy unchanged. Statement (ii) can be proved considering the Jacobian of the linearization of ODE [2]. Without loss of generality, assume that Y_i = {A, B} and the certain outcome at the SRE is $\boldsymbol{y}_{\mathbf{SRE}}$ = (A, A). Choose $p_{1,\mathrm{B}}$ and $p_{2,\mathrm{B}}$ as the two independent components of the system, so the SRE is $[p_{1,\mathrm{B}}, p_{2,\mathrm{B}}] = [0, 0]$. The Jacobian J at the SRE is then as follows:

$$J = \begin{bmatrix} l_1(\delta(s_1(\mathrm{B},\mathrm{A})) - s_1(\mathrm{A},\mathrm{A})) & l_1 \cdot \delta(-s_1(\mathrm{A},\mathrm{B})) \\ l_2 \cdot \delta(-s_2(\mathrm{B},\mathrm{A})) & l_2(\delta(s_2(\mathrm{A},\mathrm{B})) - s_2(\mathrm{A},\mathrm{A})) \end{bmatrix}$$

$$\text{where } \delta(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

It is then straightforward that if $\boldsymbol{y}_{\mathbf{SRE}}$ = (A,A) is a mutually satisfactory (s_i(A,A) > 0) strict Nash equilibrium (s_1(A,A) > s_1(B,A); s_2(A,A) > s_2(A,B)) and at least one unilateral deviation leads to a satisfactory outcome for the non-deviating player (s_1(A,B) ≥ 0 or s_2(B,A) ≥ 0), then the two eigenvalues of J are negative real, so the SRE is asymptotically stable. █

Notes to extend the theoretical results to populations of players. All the lemmas and propositions in chapter 4 and this appendix can be easily extended to finite populations from which two players are randomly drawn to play a 2×2 game taking into account the following points: (1) the state of the system S_n in iteration n is the mixed-strategy profile of the whole population. (2) An event E_n in iteration n comprises an identification of the two players who have played the game in iteration n and their decisions. (3) Pure states are now associated (in the sense given in the notation of the appendix) with *chains* of events, rather than with single events. A pure state s is associated with a finite chain of events c (where every player must play the game at least once) if the occurrence of c pushes the system towards s from any other state.

Proof of Proposition 4-5. Let Θ be the *mixed-strategy space* of the finite normal-form game. The proof consists in applying Brouwer's Fixed Point theorem to the function $\mathbf{W}(\boldsymbol{p}) \equiv \mathbf{E}(\boldsymbol{P}_{n+1} \mid \boldsymbol{P}_n = \boldsymbol{p})$ that maps the mixed-strategy profile $\boldsymbol{p} \in \Theta$ to the *expected* mixed-strategy profile $\mathbf{W}(\boldsymbol{p})$ after the game has been played once and each player has updated her strategy $\boldsymbol{p}_i$ accordingly. Since the mixed-strategy space Θ is a non-empty, compact, and convex set, it only remains to show that $\mathbf{W} : \Theta \to \Theta$ is a continuous function. Let $\mathbf{w}_i(\boldsymbol{p})$ be the ith component of $\mathbf{W}(\boldsymbol{p})$, which represents player i's expected strategy for the following iteration. Therefore:

$$\mathbf{w}_i(\boldsymbol{p}) = \sum_{y \in Y} \Pr\{\boldsymbol{d}\} \cdot \boldsymbol{r}_i^y(\boldsymbol{p}) = \sum_{y \in Y} \Big(\prod_{i \in I} p_{i,y_i}\Big) \cdot \boldsymbol{r}_i^y(\boldsymbol{p})$$

Since all $\boldsymbol{r}_i^y(\boldsymbol{p})$ are continuous for every $\boldsymbol{y}$ and every i by hypothesis, $\mathbf{W}(\boldsymbol{p})$ is also continuous. Thus, applying Brouwer's Fixed-Point theorem, we can state that there is at least one $\boldsymbol{p}^* \in \Theta$ such that $\mathbf{W}(\boldsymbol{p}^*) = \boldsymbol{p}^*$. This means that the *expected change* in all $(p_{i,j})^*$ (probability of player i following her jth pure strategy) is zero. ∎