3 Problems and Properties

3.1 Examples of Common Usage

I will introduce this section with some concrete examples, and recount some common sense perspectives on them. These will be referred to later and used as a motivation for the forthcoming discussion.

3.1.1 A case of nails

It would be very unusual for someone, on opening up a large chest of nails, to exclaim “Oh, how complex!”. This indicates that the mere fact that there are a great many nails is, at least in some circumstances, insufficient for the assembly be considered complex (this would not prevent someone modelling the forces inside the pile[17] considering it complex).

3.1.2 Writing a thesis

A thesis is easier to write with a complex tool (a word processing package with advanced features), than with a simple one (a ballpoint pen). This illustrates that the complexity and difficulty of a process can not be naively associated. Here it is the complexity of the *problem* that is associated with the level of difficulty, and the tool chosen forms part of the framework for tackling this.

3.1.3 Mathematics

If anything is intuitively complex then abstract mathematics is. This complexity seems inherent in the subject matter despite the fact that questions of mathematics can be purely formal. Even when the mathematician has complete information about the rules and components of a problem, producing a solution or proof can still be very difficult.

3.1.4 A gas

The macroscopic physical properties of a gas are fairly simple. Even though we know this is the result of a multitude of interactions between its component parts. If we had to explain these properties via an explicit and deterministic model (i.e. this particle collided with this one which…), this would be a formidable task. If we take as our base a

17.That this is a difficult modelling problem see the chapter in [91] on sandpiles level of description that ignores a lot of the detail and ascribes this to an unanalysed randomness, then the task is considerably easier.

3.1.5 An ant hill

In this example the interactions between the parts (i.e. the ants) are non-trivial, so an adequate model would probably have to include details on this. Now the task of explaining the macroscopic behaviour, given a model of the interacting parts, is challenging irrespective of whether the macroscopic behaviour is simple and predictable or not. This is the case, even if the ant colony looks very simple to someone else who views it as a rather stupid and reactive single animal that just happens to consist of physically separate parts.

3.1.6 A car engine

Consider three views of a car engine[18].

The first view is of the engine as a single inexplicable unit, a sort of glorified random number generator - it either works or it doesn't. No explanation is required or deemed relevant, its running is a matter of irreducible luck. If it does not start you ring a mechanic who magically starts it for you. This is the engine as a simple, if malevolent, entity.

The second is a view involving a partial knowledge of the engine. The parts are roughly identified as well as some of the interactions. However these interactions, taken together, are far too complex to understand. If something goes wrong, you can look inside the bonnet and try to identify the cause. Simple experiments in terms of fixing it are possible. Sometimes, with luck, this seems to fix it. Unfortunately, this action often has unforeseen consequences and causes greater long-term damage. When fixing it is beyond this level of understanding, the mechanic is called, who must be (from this viewpoint) a craftsman of deep skill and have a sophisticated understanding of the machine. This is an engine at its most complex.

The third view is (hopefully) that of mechanics. They have an understanding of the decomposition of the engine into functionally near-independent parts. They can use this model to systematically analyse the problem by eliminating possible causes, until the search narrows down to the actual cause. They can then fix this cause, having a good idea

18.This subject was suggested by my supervisor as a pertinent example, John Chidgey.

of the possible side-effects of this action. This is the engine as manageable complex, due to the appropriateness and utility of the mechanic's model of it.

3.1.7 A cell as part of an organism

When considering a cell and the organism of which it is part, from the same frame of reference, it seems obvious that the cell is simpler than the organism. This is true irrespective of the complexity of the cell, for other cells can not take away from the complexity already there in the cell.

Such a comparison is not so obvious if both are compared from within different frameworks. A cell under a microscope might seem much more complex than the potato it was taken from, viewed with normal vision.

3.1.8 Computer programming

As with the *writing a thesis* example in section 3.1.2 on page 44, the choice of programming language can greatly effect the complexity of a programming task. A language like BASIC may be relatively easy to learn, but difficult to use for a large programming task. Compare this to a language like Smalltalk which has a sophisticated object-orientated organisation allowing a high degree of modularisation and flexibility and a large set of pre-programmed classes that you can adapt and re-use. This takes considerably more time to learn, but can then make the task of programming much easier.

3.2 Complexity as a Comparison

In the common sense world complexity is not so much measured as compared. "a computer is more complex than a calculator", or "although the rules governing unemployment benefit are more complex than those concerned with income support, those covering disability benefit are the worst" are two examples of this.

Such comparisons are easiest to see when you are comparing a sub-system with the whole, from within the *same* frame of reference, as in the cell example above (section 3.1.7 on page 46). In other cases it is not at all clear, as when comparing the complexity of a computer and a poem. Here we have no natural common basis from which to make a comparison. We could artificially construct such a basis but there would be little likelihood that this would agreed upon by others. Without an appropriate framework within which to represent both, any such judgement of relative complexity would be arbitrary.

3.2.1 The emergence of life

One paradigm of emergent complexity is the appearence of life. Most people would say that the complexity of the biosphere has increased with the emergence and development of life. That is, if one compares the solar system as it is now and how we think it was 5 billion years ago, then the obvious conclusion is that it has become *more* complex in important respects. The fact that the subsystems which exhibit the change are not great in terms of mass or volume does not significantly alter this judgement.

3.3 What the Property of Complexity Could Usefully Refer to

I this section I will argue that regardless of whether our models reflect reality in other respects, complexity is most usefully attributed to the *descriptions* of our models and is only projectable back onto natural phenomena when constraints on our choice of models make this coherent.

One could attempt to distinguish what complexity was an *intrinsic* property of, and thus argue that complexity was an extrinsic property of natural systems but an intrinsic property of model descriptions. Such a route is fraught with difficulties and would not further the purpose of this thesis, which is pragmatic. I am concerned here with developing a characterisation of complexity that *usefully* captures our intuitions of the concept.

3.3.1 Natural systems

I will argue that complexity is not a property *usefully* attributed to natural systems. I list the arguments below.

1. *Estimates for the lower bounds of the complexity of natural systems can always be increased by the inclusion of more aspects of the system*

By increasing the level of detail considered, a lower bound for the complexity of almost any natural system can be arbitrarily increased until it is well beyond our means of representation, understanding and modelling ability. There is no apparent upper bound to how complex things can appear. It seems that most things have the potential to be arbitrarily complex, just dependent on the number of aspects and the level of detail one is willing (or able) to consider.

This can be seen as a consequence of our intuition that a sub-system is less complex than the whole. By expanding the context of consideration (or equivalently using a more general framework) one includes more complex sub-systems. This forces us to conclude that the complexity of the whole system is greater than these sub-systems (unless one is allowed to ignore the effects of these sub-systems using appropriate modelling assumptions).

For example, a brick is a fairly simple object if you consider only its macroscopic properties but is much more complex at the level of its component particles. This particulate complexity is averaged out at the macroscopic level at which we usually relate to them, so we can usually abstract from the details, but if you are insisting on an entirely objective basis (what ever that would mean) then you have to allow for the inclusion of this particulate complexity. The complexity further increases beyond our comprehension when we consider such a macroscopic object, such as a brick, at the sub-atomic level of detail.

2. *Estimations of the practical complexity of natural systems can change critically when the framework in which they are considered is varied*

Changes of goal, language of representation, aspect, and scale can all greatly effect the practical complexity of natural systems. The point above is a result of the potential to include more detail by changes of scale and generalising so as to include more aspects of the system under study.

In the ant colony example (section 3.1.5 on page 45) whether your goal was merely predictive of behaviour at a purely macroscopic level or was seeking to explain this macroscopic behaviour explicitly in terms of the behaviour of individual ants, affects the practical complexity of the modelling task.

One example of how the language of representation can critically affect the complexity is in the representation (or exclusion) of elements as noise. Often ascribing some parts of some natural phenomena to noise can allow the drastic simplification of the representation of a natural system. For example in the gas example above (section 3.1.5 on page 45) being able to assume that the detailed movements of the particles in a gas are random allows a model of the macroscopic properties to be related in a simple way to the microscopic ones.

Thus intuitive assessments of the complexity of systems, often differ far more with changes across frameworks than across changes of subject matter.

3. Attributing complexity to natural systems does not help explain the existence or process of simplification

The ascription of complexity directly to natural systems also makes an account of simplification difficult. One would be forced to judge all equivalent models of a natural system as equally complex. Thus an account of planetary orbits using an infinite series of epi-cycles would be as simple as one using ellipses. Similarly there would be no simplification of mathematical systems if complexity was a property of the *content* of these systems, since all we would be doing is changing their description.

One method of simplification (discussed in section 5.7.5 on page 124) is to trade-off complexity for the specificity or accuracy of the model. A less specific model is one which has either narrower conditions of application or else its predictions are over a broader range – for example, the use of fuzzy-logic and fuzzy-set theory has been suggested as a means for dealing with complexity in some situations by Zadeh [488]. A less accurate model is one with a greater level of error with respect to the data model – for example, one might decide that some variation in the data could be attributed to noise so that accepting a greater level of error might result in a much simpler model. The 'stochastic complexity' of Rissanen [378] is an attempt to find a principled trade-off between error and model complexity.

In fact simplification and elaboration (for the want of a better antonym) are frequently what we are concerned with when we talk about complexity – complexity has somehow arisen and we need to deal with it. Some strategies for simplification are discussed in section 5.7 on page 120.

4. Our intuitions about the complexity of natural systems can be nicely accounted for by associating them with the complexity of the systems 'best' model

If we have a natural system which is producing what seems to be random data as its output (where we know that this is not attributable to a separate and discountable source of noise) it can still be sensible to say that this data is simple on the grounds that it is random.

Here the complexity of the system has been taken from its most appropriate model, which in this case is *not* the most descriptively accurate but is a less specific model. A probabilistic model is more appropriate in circumstances where we know the detail of the individual sequence is not relevant. We lose the possibility of completely accurate predictions for a considerably simpler model.

In cases where our models of these systems are considerably constrained (by nature or by practice) we are sometimes in the fortunate position of only having one candidate model in which case it is safe (in that context) to project this model's complexity upon the natural system. As Cartwright puts it:

"It is precisely the existence of relatively few bridge principles that makes possible the construction, evaluation and elimination of models … it strongly increases the likelihood that there will be literally incompatible models that all fit the facts so far as the bridge principles can discriminate." [84] p.144

One of these principles is surely that we exclude models that are prohibitively complex. Reinberger recently said:

"Reduction of complexity is a prerequisite for experimental research." [374]

All of the above difficulties come down to the same nub: if natural systems do have complexities, then they are unmanageablely large. Thus at the moment, and quite possibly absolutely, it is not useful to try to do this. If we choose only one aspect and one scale, we are no longer dealing with the complete object, but an abstraction of it.

This can be traced to why we *ever* consider properties to be of things rather than our models of them in the first place – because they can be said to have the "same" properties independent of the observer or the models[19] (e.g. mercury or thermocouple models of temperature) - there is no evidence that this is true of complexity judgements.

There are several possible arguments against this. I consider these below.

1. *Some complexity comparisons concerning natural systems are objective.*

An example is "An amoeba is objectively simpler than a human". There are two ways of interpreting this argument, firstly as a variant of the cell example (section 3.1.7 on page 46), i.e. that an amoeba is simpler than a human *however you look at it*, and secondly

19.This is a simplistic account – accounting for why we call a property the same from different frameworks is not straightforward.

as an implicit call on upon a privileged (and hence common) framework, i.e. there is *one* sensible framework to use and within this the amoeba is simpler.

The first argument assumes that the comparison is valid, *regardless* of the framework. This is an immensely strong assumption, one that seems to draw its strength from an identification of the amoeba with a human cell[20] and invoking the sub-system property that I noticed in the cell example. Otherwise it would seem possible to choose a framework where the amoeba differed substantially from the cell (its method of encapsulation and subsequent digestion of food?), where the judgement of relative complexity was not so clear. If one then argues that this is an inferior or more specific judgemental framework, this would bring us to the second interpretation.

The problem with arguing for a uniquely (or even relatively) privileged framework is that of its justification, given that it does not allocate impractically large complexities to almost everything (or that a framework revealing *more* complexity is not always better). Also that in practice there are always pragmatic choices of factors like scale, so that a privileged framework is no use for actual complexity judgements. Finally the very identity of many things seems inextricably linked to which aspects you are considering (e.g. ‘society’).

2. *Claiming that complexity is not a property of natural systems is just a category mistake.*

It could be argued that even if it is admitted that complexity comparisons can only be meaningfully via our models, the complexity refers to the natural systems themselves, and that it is only due to our limitations (in particular our understanding) that we see different complexities from different viewpoints.

This is a possible standpoint, but it is hard to see how this could be then applied in practice without *in effect* attaching the property of complexity to the models rather than the original systems, as otherwise in practice the complexity of such systems would change arbitrarily depending on the model chosen. Such a view would be comparable to that attributing primality to sets of real objects rather than numbers. One could characterise number by equivalence classes of things, i.e. five is the class of all possible

20. Or alternatively an identification of the functional organelles of an amoeba with the corresponding functional parts of a human, though this does not alter the argument.

sets of five objects. This still would mean attributing a property such as primality to something other than the things themselves. It is difficult to see this as anything other than a contrivance.

3. *Attributing complexity only to model descriptions would make objective complexity judgements impossible.*

If a framework is agreed upon then the complexity of something can be objectively determined by different observers with respect to this framework. So once this framework is established complexity judgements can be consistently made irrespective if who is doing it as long as they keep within the rules that the framework entails. This is not so different from many other ‘objective’ judgements and facts – such frequently rely upon such contextual bases (what Suppe calls the ‘disciplinary framework’ [428]).

For example in the physical and mathematical study of chaotic systems a framework which implicitly disregards some level of detail as noise is so familiar that it has become a background assumption (see the section on Noise on page 206). Researchers in this field seem to uniformly agree that a perfectly disordered system is simple (like the gas example, section 3.1.4 on page 44, where this uniform randomness makes it a candidate for a level of simplicity akin to a crystal[21]). Of course, it is advantageous to make these background assumptions explicit, so that if necessary they can be checked.

4. *There are real causes of complexity*

This may be true but not a compelling argument for the use in considering complexity as pertaining to natural systems. It derives from a confusion between accounts of how we are to characterise complexity and what can *cause* it. To take an analogy heat can be *caused* by a variety of forms of energy which *are* themselves not represented as heat.

In the end pragmatic considerations prevail; it is *useful* to attribute a property to a natural system when this is largely independent of our models of it but more useful to consider it a property of the models if it is not (e.g. beauty is partially attributed to the

21.For a classic account of this see Grassberger [194].

beholder, because is known to be at least partially culturally dependent - 60's tower blocks - Miss World competition). As Suppes said:

> "*We can only hope to have a concept of complexity for explicit models of the world and not for reality itself or even small parts of it.*" [433].

3.3.2 The interaction of an observer with a system

In response to these problems many authors (e.g. Casti [87]) have stressed that complexity only makes sense when considered as relative to a given observer[22]. Thus they put an observer into the picture which controls or otherwise affects the target system, and then observes (or is effected by) that system. This establishes a split between the "system complexity" and the "observer complexity" (see figure 12). The system complexity is the complexity of the system w.r.t. the observer and the observer complexity is the complexity of the observer w.r.t. the system.

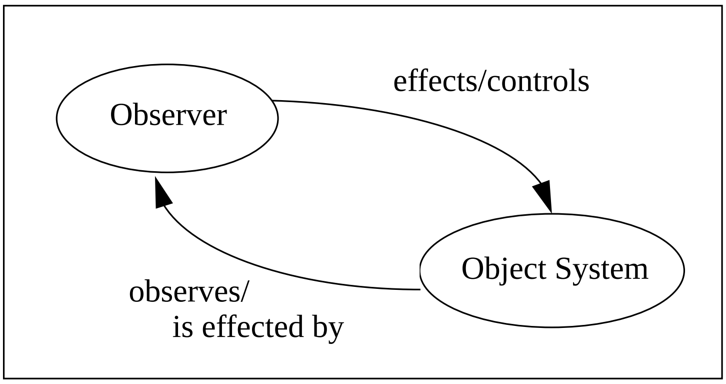


Figure 12. Observer–system pair

You can look at this analysis in two ways: extrinsically and intrinsically. In the first, there is no essential difference between the two systems - from an external point of view one just sees two systems interacting. In the second, we are describing the situation from the observers point of view.

If you take the extrinsic interpretation, then there is nothing special about the observer system. The 'loop' in the diagram (figure 12) is misleading as there may be multiple parallel and re-entrant interactions between the two systems and between

22.This distinction is basic to systems theory, going back to Ashby [23, 24].

different sub-systems of the observer and object system (as in figure 13 on page 54). In this case we are merely dealing with one system composed of the pair of object system and observer. The observer and object may or may not be easily separable or have effectively separate identities. All we are dealing with is a particular decomposition of a single system. So the observer and system complexity become merely the complexity of a sub-system with respect to the rest.

If you take the intrinsic view then there are still some problems with this approach, namely:

1. *It is still difficult to ascribe useful meaning to the complexity of the observer w.r.t. the system unless the system is an observer too, otherwise the observer is itself unobserved and so the complexity undefined in the same way as with natural systems (section 3.3.1 on page 47).*

2. *The complexity of the system w.r.t. the observer will still vary according to which aspect of the observed system is being considered by that observer.*

The observer could be taken as to refer to a particular identified individual. This individual could use several different internal representations to model the system and interact differently according to each at different times. If these act in some way such that they can be considered together as a composite model then you are back in the situation illustrated in figure 12 on page 53. Such a scenario is illustrated in figure 13.

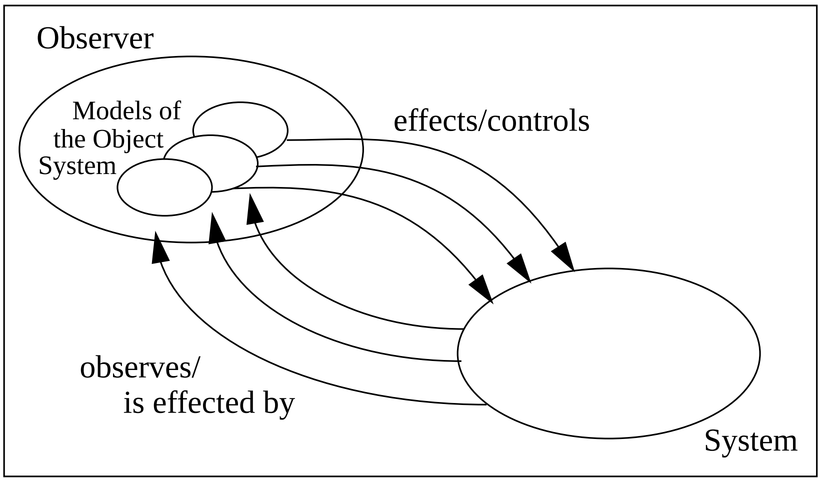


Figure 13. An observer-system pair with multiple models

Either the separate models can be said to be interacting with the system (through the observer) or the internal models can be said to form some sort of a composite model. If, in the former case, the separate models would not qualify as separate observers, then problems of complexity being changeable according to the particular viewpoint chosen reappear. In other cases the situation can be further abstracted as to be between the model/representation and the system.

So which ever way you interpret the observer-object system analysis, you come back to the same problem of practical reference to an embedding framework drastically effecting the effective complexity of the observed model.

3.3.3 Patterns

Next I will argue that complexity is not a useful property to ascribe to patterns[23]. In other words, to make a meaningful judgement as to the complexity of a pattern you need a syntax.

Consider two patterns generated by a random process; how can you judge them as differently complex? One may seem to be more meaningful and the other not but this is maybe just happenstance, it may be that they are both equally probable and generated by the same process. It is only through the interpretation of some process that they may be said to differ (for example by compression to a minimal length Turing machine that would output it, see section 8.2 on page 136). The trouble is that the same pattern can be decomposed or interpreted in many different ways. Each decomposition might give a different picture of its complexity. In Appendix 2 (section 9.1 on page 164) I show that given some reasonable assumptions that there is no non-trivial complexity measure on one-dimensional patterns. This counter-example relies on the possible multiplicity of decompositions - it contrasts with the demonstration of complexity measures upon a structured language given similar assumptions.

The usual manner for dealing with the complexity of patterns is not to compare the patterns themselves, but to compare the respective sets of rules for generating these patterns (even if these rules are merely guessed at). This explicitly provides them with a syntax, and hence they become more than a pattern.

23. By a pattern, I mean data that is ordered (typically in time or space) but is not restricted by a combinatorial syntax.

3.3.4 The modelling relation

Rosen [385], Casti [89] and others, have formalised the process of modelling in terms of a “modelling relation”. This was explained in section 2.4 on page 38. The intention is that a chain of causation in the natural system is modelled by first encoding the initial conditions into the formal system, then following the chain of formal causation and finally mapping this back into the natural system. This is illustrated in figure 11 on page 41.

If the modelling relation commutes the formal system is said to be a *model* of the natural system. Rosen then characterises “complexity” as an attribute of a natural system, if there are many such “inequivalent” models (see section 6.7 on page 131). Casti quantifies this as the number of inequivalent models (section 8.29 on page 152).

In order for there to be many such models, there must be many possible encodings of the natural system. In order to allow for the existence of several such encodings, the whole must lie within a larger framework. In order for “inequivalence” to be well defined, this larger framework must *itself* be sufficiently defined. Finally, this framework must be limited in scope, otherwise all natural systems would trivially have many models and thus be “complex” in this sense, in which case it would not be a useful property *of* such “natural systems”, as it would be coincident with the property of *being* a natural system. It is unclear in this case that *any* natural systems would be counted as simple by this criterion.

Thus, in order to be meaningful, this approach to attributing “complexity” to natural systems, must be defined relative to a larger framework. To the extent that this framework is well-defined, then this approach will be also. Some of the advantages and difficulties of this approach will be discussed in section section 4 on page 72.

3.3.5 A model with respect to a specified framework

In view of the above analyses I contend that complexity, if it is to be a useful attribution, needs to refer to a model relative to the modelling framework. That this is a possible or useful approach will be demonstrated in section 4 on page 72, where I will give a working definition of complexity and apply it to some examples.

3.4 Some Unsatisfactory Accounts of Complexity

Here I will briefly discuss some ideas that are frequently conflated with complexity, but which are, at best, very weak models of it. In each case conflating the concept with complexity will mean the loss of a useful analytic distinction. In doing so I hope to motivate some of the details of the next sections concerning the desirable properties of a complexity measure and its definition.

3.4.1 Size

There is clearly a sense in which people use “complexity” to indicate the number of parts, but it seems rarely used *just* to indicate this, as was shown in the case of nails example (section 3.1.1 on page 44). Contrast this example with that of an intricate (mechanical) watch, where the appellation of complex might be more appropriate.

The difficulties raised by size are real, but can be weak when compared to other such difficulties; it is likely that simple problems such as size can be dealt with in simple (albeit possibly expensive) ways[24].

Intuitively there are large but simple systems, such as a book of random numbers, or the list of facts behind ‘Trivial Pursuit’ questions (where there is no intended relevance between the questions other than the categories of sport, entertainment etc.). Doubling the size of either would not significantly increase their complexity.

Size seems not to be a sufficient condition for complexity. On the other hand a certain minimum size does seem to be a necessary condition, it is very hard to imagine anything complex made up of only one part. This minimum size can be quite small, there is a Turing machine defined with only five states that comes to a halt after exactly 23,554,768 steps and the task of finding the maximum number of steps a seven-state Turing machine could definitely halt in has been described as “hopeless” by Machlin [303].

If a system is broadly symmetrical in terms of the relations between its parts (for example in a peer organised computer network), then size might be a good indicator of complexity, but otherwise the structure of the system might have a far more critical effect.

24.This is not to deny that if there a size limitation is a critical factor this may not qualitatively change the situation to a complex one (as Anderson [13] points out).

Broadly size seems to limit the potential for complexity, rather than determine it. See section 8.38 on page 157 for examples of size used as complexity.

3.4.2 Size of rules

A variation on a purely sized-based measure would be to base a measure on the size (in some sense) of the *description* of a system (in some language). This produces far more acceptable results, for example the book of random numbers would be judged in its complexity by the size of rules that generated it, rather than the size of the results.

However this still has many of the same problems as simple size measures have. One can imagine a case where a system was generated by hundreds of independent rules (e.g. 1001 unrelated tips about how to succeed). Thus the interaction between the rules must be still important as to the overall complexity. Of course, it is hard to imagine examples in natural language where there is absolutely no relevance relation between separate descriptions, due to the involved and intricate nature of language. This does not stop it being the case that the amount and structure of this interaction or relevance between the parts of the description will effect the complexity of the result.

One can imagine further elaborating this sort of approach by then taking the description of the description of the pattern and thus further improving this way of measuring the complexity of the original object. This improves the suitability of this sort of measure for complexity in that it encodes more about the structure inherent in the object, but in this case one is abstracting further and further from the basic model of size as a measure. The complexity of the original object is now far more encoded in the language of description than the eventual associated size. On the whole it seems that the more a judgement of complexity encodes *structure* rather than mere *size*, the more it fits our intuitive picture of complexity. This puts size measures in their place, they encode only the most basic property of structure - the number of parts the structure is to be built from. Thus if simple size could be seen as a measure of complexity it would be amongst the weakest possible of such measures. See also section 8.39 on page 157 on size of grammar for examples of this approach.

3.4.3 Minimal size

This brings us to a further elaboration of size as a complexity measure: that of the *minimal* descriptive size in some language or by some compression process (see also section 8.24 on page 149). The most commonly used form of this is “Kolmogorov” or “Algorithmic Information” complexity (AIC) - the size of the smallest Turing machine that would generate the object pattern (see section 8.2 on page 136).

AIC represents the high end of minimal descriptive size measures - those using the most powerful compressive machinery (alternatively those using the most expressive language). The compressive machinery is so powerful that it gives a highly unpredictable measure. For example the second 1,000,000 digits of Pi are indistinguishable to most people from a list of 1,000,000 random numbers, but the first would have a small AIC and the second would (almost certainly) have a large AIC[25]. Here the relation of the size of the minimal program and the object are only marginally related - a fact that is starkly illustrated by the fact that the AIC size is uncomputable, in general, from the original object. I argue later (section 4.3.4 on page 84) that AIC is a more appropriate measure of information.

At the other end of the scale, if the compressive machinery is very weak (the associated language is inexpressive), then this is not much better than a measure based on simple size, since the size of the object and the size of the minimal description will be highly related.

Measures based on minimal descriptive size could be an acceptable complexity measure if the relevances between the parts of the original that the compressing machinery exploited (i.e. the redundancies) were the appropriate ones for the system and purpose in mind. In any case this is now primarily no longer an issue of size but structure.

3.4.4 Processing time

It is hard to imagine a difficult task that can be done without some time spent on it, either in execution or preparation. Thus the complexity of a task can come to be associated with the amount of processing time it requires (see section 8.47 on page 162 for examples).

That this is a probable consequence of complexity and not a sufficient condition for it can be seen in the existence of intuitively simple but time-consuming tasks. Trying out all the possible colorations of a checker board of a certain size, is a simple task, but given m colours and a board of size n there are m^{n^2} possibilities, so trying out the colouration of

25.It is overwhelming likely that the AIC of a random string is not less than the length of that string [284] a 20 by 20 board with 10 colours would be far above the maximum possible computation performable by the whole universe since the beginning of time [75].

Secondly, like size, if processing time is a measure of complexity (see section 4.3.3 on page 84) then it is a fairly weak measure, relatively unreflective of the structure of the task. Processing time may sometimes be an insuperable difficulty in practice but it is a simple difficulty, solvable by simple means (more time)[26]. Personal experience tends to suggest that the process of writing a program often presents a more fundamental level of difficulty than the time it takes to run.

Like size the application of processing time as a measure of complexity can be applied at a more abstract level, for example, to the time taken to write a program to preform a certain task. This brings this sort of measure closer to the intuitive meaning of complexity, but at the cost of the direct appropriateness of the "processing time" measure. In the above example, in order for the programming time to be well-defined, the programming environment and methods would need to be specified, then the complexity of the task is more encoded in that environment than in the programming time taken.

3.4.5 Ignorance

Complexity is a major cause of ignorance - if a problem is complex, we often do not know how to solve it. Ignorance is sometimes a cause of complexity - due to our ignorance we can choose an inappropriate framework for considering a problem and this makes solving it more complex. It is thus tempting to associate the two. I will argue that information (or the lack of it) and complexity are more usefully considered as different aspects of a problem.

Firstly, there are some tasks where we seem to know everything, but they are still complex. This depends upon your scope of "everything". In many puzzles and games one has complete knowledge of the rules and situation, but finding the solution is still complex. If one broadens the scope to include knowledge of the solution (or optimal playing strategy), finding the solution may be no longer difficult, but implementing it may be. Some games and puzzles have no *easy* solution (e.g. chess).

Secondly, there are situations where increasing the knowledge about the solution does not make it simpler. Imagine a task of designing the best program to check the code

26.This does not prevent questions *about* processing time being complex.

of another program to see if it will halt, given a particular input. We know that there are no techniques that do this in general [443]. Knowing this does not make the task less complex!

So although ignorance and complexity can interact[27], they should be considered as separate sources of difficulty - requiring different types of solution. Separating these two factors will also allow the study of their interaction - which could be very useful. See section 8.15 on page 144 for examples of this approach.

3.4.6 Variety

Simple systems themselves do not display great variety[28]. Thus complexity can be associated with variety. An animal with a greater variety of shapes of invertabra is said to be more complex (in this respect) than one with fewer (e.g. McShea [316]).

An immediate increase in variety, can however, be accompanied by a decrease in complexity. A list of prime numbers up to a certain size is not less complex than a list of all such natural numbers, despite the fact that there must be at least as much variety in the list of natural numbers by construction. It is implicit in this comparison that it is not the sequence itself that is really being considered, but the rules/source of it - thus the rules to generate prime numbers are more complex than those for a sequence of all numbers. So the application of variety shifts to these rules, but this does not completely avoid the problem, as a base of *all possible rules* might not produce very complex behaviour.

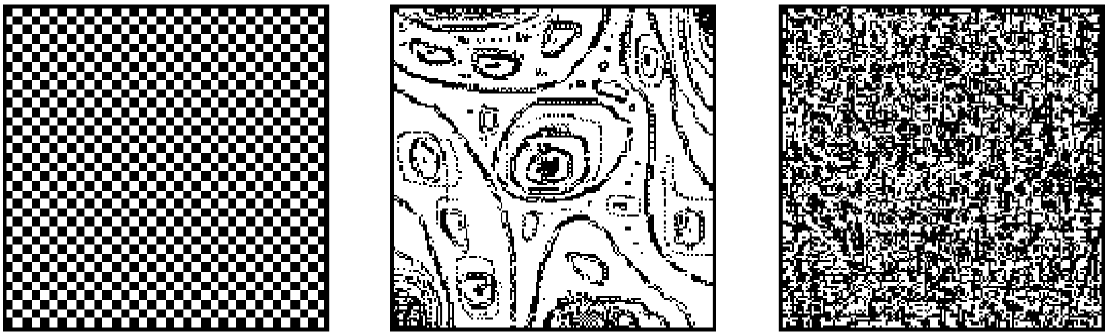
Another possibility is that you measure variety by minimal description length described above (section 3.4.1 on page 57), but then the arguments of section 3.4.3 on page 58 would apply. See section 8.48 on page 163 for examples of this approach.

27.See, for example the “problem complexity” of Waxman in [463].

28.Although categories (or systems) of simple systems do.

3.4.7 Midpoint between order and disorder

Complexity is sometimes posited as a mid-point between order and disorder. Grassberger [194] considered three patterns similar to those in figure 14 below.



1. Order **2. Chaos** **3. Disorder**

Figure 14. Complete order, chaos and complete disorder

The immediate reaction is to judge that the first and last patterns are simple and the middle one relatively complex (leading to diagrams such as figure 15 on page 62, e.g. in [297]), but this is due to the facts that our perceptions 'filter out' the complexity of the right-hand pattern and that we interpret it as *representing* a situation with *no rules*[29] (i.e. random). Thus, we are not judging these uniformly; it may well be that the right hand pattern represents *such* a complex situation that we don't recognise it. To illustrate this, consider the possibility that there may be a small version of the left-hand pattern included in the middle and a small version of the middle pattern included in the right-hand pattern (as in figure 16 on page 63).

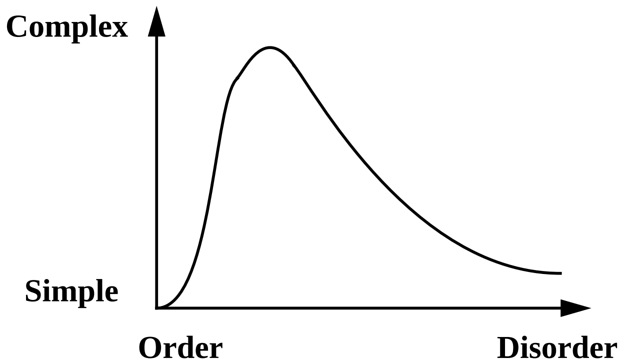


Figure 15. Presumed graph of disorder against complexity

29.Or a single rule involving randomness. Whether this would be a complex rule would depend on whether you judge randomness as fundamentally complex or whether it is an allowed atomic rule form.

If this is the case, we are forced to judge the patterns in order of increasing complexity from left to right. We see the importance of the language of representation. If we were considering the complexity of some (assumed) rules to generate these patterns, then the original intuitions might be preserved. As noted in section 3.3.3 on page 55 the confusion comes because such patterns do not have an inherent language[30] - we have to impose one on them.

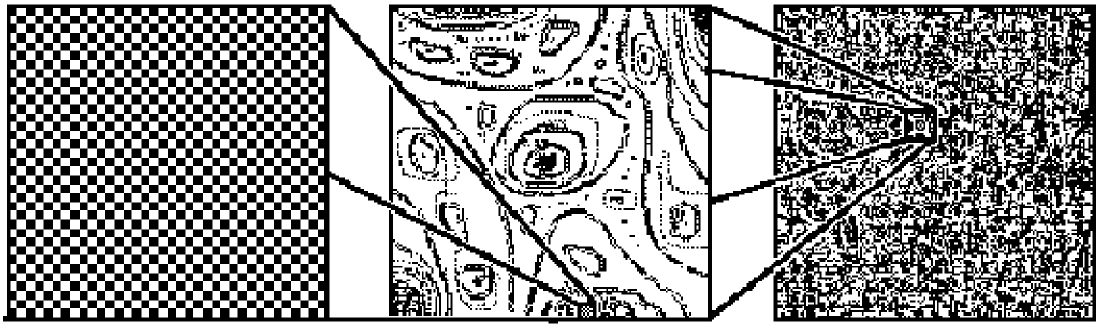


Figure 16. Possible diagrammatic inclusions

A footnote of Grassberger's is particularly revealing here. It says:

> *"Some people hesitate between the middle and right panels when being asked to point out the most complex one. But once told that the right one is created by means of a (pseudo-) random number generator, the right panel is usually no longer considered as complex." [194] (page 491).*

"Not being completely ordered" and "not being completely disordered" may be necessary conditions of complexity, but this gives us no way of comparing intermediate cases which are differently structured. The degree of variety and constraint involved in a system is only a rough measure of a systems structure, and thus is not an ideal measure of complexity.

3.4.8 Improbability

In many physical systems (particularly systems in equilibrium), complex behaviour at a macroscopic scale is very unlikely; a uniformly disordered state at the microscopic level is normal.

30.Except maybe an assumed one, which may be different for different observers.

There is a marked contrast here between the different levels of description. A simple, uniform equilibrium state at the macroscopic level hides very complex and highly disordered detail at the microscopic level. Conversely, more complex behaviour at the macroscopic level is often based upon a more ordered state of affairs at the microscopic level because a disordered state (high entropy) tends to be very stable.

In terms of information the most efficient coding that can be devised will use less information to specify or record probable events than unlikely events. In this sense a probable event gives you less information because you expected it anyway [408].

Thus a high probability macroscopic state coupled with a highly disordered (high information) microscopic state is associated with low complexity and a low probability macroscopic state with an ordered (low information) microscopic state with high complexity. Thus entropy and other probability based measures are linked with complexity.

There are also low entropy states associated with low complexity, like cold perfect crystals, so the connection between entropy (of some of the other probability/information based measures) and complexity is not straightforward, as shown by Li [287]. In some highly dissipative systems with noise, however, complex behaviour is almost certain and a lack of it would be very surprising. An example is convection in a fluid between two parallel plates with a sufficiently great temperature gradient between them [361]. Thus the association of complexity with improbability only holds for a restricted set of equilibrial systems.

Part of the problem here is in the implicit determination of the appropriate scale for the phenomena to be viewed at, i.e. what level of detail is considered meaningless "noise" (see the section on Noise on page 206). The improbability of a complex state is entirely dependent on this scale, for without such an approximation every state is equally probable and meaningful.

Finally, complexity measures based on probability do not directly apply to purely deterministic systems without some form of course graining applied to them as above.

See section 8.22 on page 148 for examples.

3.4.9 Expressivity

If a type of statement can express a lot of things, i.e. it has expressive power, then it is likely to contain some more complex statements than a type which has a more restricted range. It is certain to include more such complex statements than a sub type could. You can not describe some complex phenomena without sufficient expressive power, and the theory of more expressive types is, in general[31], more difficult than the theory of less expressive types. So an indication of the complexity of a statement is the least type it belongs to, i.e. the minimal expressivity needed to encode it. Examples of this are Logical Complexity (section 8.20 on page 146), and Kemeny's measures (section 8.18 on page 146).

The effect of this analysis can depend on how fine-grained the type classification is. If the difference between adjacently expressive types is large then this analysis will not specify much about the expression, as many different possible expressions will be possible within each classification, all differing in complexity, allowing the possibility that an expression of a higher type could be intuitively simpler than that of a lower. This is especially likely if the hierarchy of expressivity only encodes one aspect of the expression’s potential complexity. For example comparing arithmetic expressions over the integers and reals, the solution set of $x^2 = 2$ can only be expressed in the reals, but is much simpler in every other respect than many equations over the integers[32], similarly the task of computing the number of zeros of an equation is computable over the reals but not over the integers [128].

If the analysis is fine-grained and fairly complete in its coverage of the potential aspects of complexity in an expression, then this comes close to a language of specification to which the complexity of the expression can be judged.

3.4.10 Dimension

The number of dimensions a model *requires* is some indication of its complexity. Simple systems can often be depicted with one or two dimensions, whilst if a model needs many dimensions, it might indicate that the relationship between these are necessarily

31. This is not always the case, for example compare the theory of real and integral arithmetic, many of the problems (e.g. uncomputability) disappear when you move to reals.

32. As in the equation constructed in Chaitin [102], where the solution set is, in a real sense, unformalisable.

complex, in that they can't be reduced to fewer. Cognitive complexity (section 8.5 on page 139) and the dimension of the attractor in chaotic systems (section 8.9 on page 141) use this approach.

The number of dimensions is essentially a limitation on the expressiveness of the possible formulations. This is thus really a sub-case of complexity as expressivity (section 3.4.9 on page 65).

Dimension is thus at best a coarse limitation on one aspect of a formulation's complexity. For example, to characterise the full range of cylinder shapes you need two dimensions: height and radius, but there may be no relation between the two; to characterise the cylinder shapes of fixed aspect ratio, such that their heights are prime numbers (in some unit of measurement) takes only one dimension.

3.4.11 Ability to surprise

It is difficult to model complex systems, so it is likely that any model we have is incomplete in some respect. If we have come to rely on this model (for instance when the system has conformed to the model for some time or under a variety of circumstances) and the system then deviates from that model, then we are surprised. The ability to surprise is not possessed by very simple and thus well-understood systems, and consequently comes to be seen as an essential property of complex systems.

This is not useful as a complete categorisation of complexity for several reasons. Firstly, it is only relative to the sophistication of the model and our reliance on it. Secondly such surprise could be the result of other things like simple ignorance or error in model formulation. For example, we do not know whether the processes behind some apparently unpredictable quantum effects are due to a complex internal chaotic process or a simple fundamentally random one. Chaos theory tells us that completely deterministic mechanisms can generate sequences of data that are statistically indistinguishable from random sequences. This poses the question of whether such "random" processes are, in fact, just very complex ones and hence merely reflections of our paucity in modelling (see also section 6.2 on page 126 and the section entitled Order and Disorder on page 203).

3.4.12 Logical strength

One can think of statements as "holding within themselves" their logical consequences, which are revealed when one applies the machinery of a logic's proof theory to them. This picture is formalised when one identifies a formal proposition with its logical theory. Combining this picture of propositions with the principle that systems are at least as complex as their subsystems leads to the conclusion that logically stronger propositions are more complex[33]. Löfgren [293] uses logical strength as the basis for a measure of 'interpretative complexity'.

Viewing propositions like this compels one to accept that either the logic has been effectively simplified by identifying all equivalent propositions with the same theory or that each theory has a complex (and often infinite) set of labels. Propositions would essentially be considered only from the point of view of their proof theory and not from other frames of reference (e.g. their syntactic structure). This is especially obvious when the contradiction $p \wedge \neg p$ is considered, this is the strongest possible in classical propositional logic and so by this characterisation must be the *most* complex!

3.4.13 Irreducibility

Irreducibility is a source of complexity. A classic example is the three body problem in Newtonian mechanics, where the goal is to solve the equations of motion of three bodies that travel under mutual gravitational attraction. This is analytically unsolvable and hence is qualitatively different from any reduction to several separate 2-body problems[34].

To characterise complexity as irreducibility is too extreme, for two reasons. Firstly, this would mean that many formal systems would be counted as simple (see the example in section 3.1.3 on page 44). Secondly this would rule out many meaningful comparisons of complexity, for example if comparing three systems (primordial mud, a simple organism, and us), one would be forced to categorize two of these as equal in complexity (this is also the case with the second example in the "Complexity as a comparison" section above (section 3.2 on page 46).

If irreducibility is to be allowed degrees and can be meaningfully defined so that formal systems do not necessarily come out as simple then these problems are mitigated.

For examples see section 8.17 on page 145.

33.Or, at least, not simpler.

34.The five-body problem is even worse: five bodies travelling at finite speed initially can interact so that they all disappear to infinity in a finite amount of time.

3.5 Complexity is Relative to the Frame of Reference

Repeatedly, in the analyses immediately above, we saw that the effective complexity depended on the framework chosen form which to view/model the system of study. This framework is very close to what Suppe called the ‘disciplinary matrix’:

> *“The disciplinary matrix contains all those shared elements which make for relative fullness of professional work, models, ontological commitments, symbolic generalisations, a language, with meanings specific to that community, some interpretive symbolic generalisations, and so on.” [428] p.495*

In this subsection I will highlight some of the aspects of this framework in order to prepare the ground for the definition of complexity in the next chapter.

3.5.1 The level of application

In the section on complexity as the midpoint between order and disorder (section 3.4.7 on page 62), we saw how the level of description greatly affected the interpretation of its complexity. When we looked at the patterns themselves we came to a different conclusion than when we were considering possible rule sets to generate those patterns. This is also inherent in many of the examples (e.g. the gas example in section 3.1.4 on page 44) or analyses above.

In fact, complexity often appears when we are seeking to cross levels. In the ant colony example (section 3.1.5 on page 45), I noted the great contrast between a macroscopic and a component view of the system; the complexity occurred when I sought to explain the former in terms of the latter. When we keep to a very similar or even unitary framework, systems are often simpler; when we sought to explain the behaviour of the ant colony in terms of both stimulus and response to the whole colony our task seemed a lot easier.

The criticality of scale in the modelling of phenomena leads Badii and Politi [36] to focus their characterisation of complexity solely on such hierarchical and scaling effects. As they say:

> *“The study of the scaling behaviour of physical observables from finite-resolution measurements appears, therefore, as an essential instrument for the characterisation of complexity”* [36] *p.249*

3.5.2 Goals - type of difficulty

In the gas example (section 3.1.4 on page 44), the complexity of the gas depended on whether we were trying to explain its behaviour statistically or deterministically (i.e. with or without "randomness").

When considering processing time as a measure of complexity, (section 3.4.4 on page 59), in the checker-board colouring example the difficulty in terms of time was more fundamental than that of program space: the AIC (section 8.2 on page 136) of all the coloured checker-boards is small, but the computational complexity (section 4.3.3 on page 84) is large. In other examples, like that of producing a random sequence, the AIC will be large but the computational complexity small, illustrating how the complexity is relative to the task in hand, which is of course relative to your goals.

Another contrast is between the difficulty in trying to find (or induce) a suitable model description compatible with set of data and that of trying to analyse a given model description in terms of the properties of its content. Models that are simple to analyse can sometimes be very hard to find.

3.5.3 Atomic parts

We saw how the base units of our descriptive framework affected the complexity of our models. This was illustrated by the gas example (section 3.1.4 on page 44) - whether we included "randomness" or "noise" as a basic (and thus unanalysed in this framework) element of our framework effected the complexity of the description.

Similarly in the ant colony example (section 3.1.5 on page 45), whether we took the individual ants as the atomic units of our description or just looked at the colony as a whole mattered greatly.

Which parts are considered "atomic" (or merely more basic or primitive) is just one aspect of the language of modelling or description, but it is important as it 'anchors' what one's starting points are to be. For example, the complexity of determining the tape that results from the action of Turing machine may be very different from determining the Turing machine that will terminate with a given tape.

3.5.4 The language of description

That the language of representation and modelling is critical to the effective complexity, is clearest in the programming language example (section 3.1.8 on page 46).

Here the language is completely explicit and makes a clear impact upon the task in hand. This example has many parallels with the mathematics example (section 3.1.3 on page 44). In mathematics there are two distinct difficulties: learning to use abstract and expressive mathematical languages and using these to solve a problem. Frequently if a problem appears insoluble the solution lies in shifting to a more powerful and expressive language to attack the problem from[35]. Toulmin goes further, he says:

> *"The heart of all major discoveries in the physical sciences is the discovery of novel methods of representation, and so of fresh techniques by which inferences can be drawn – and draw in ways which fit the phenomena under investigation. The models we use in physical theories, … are of value to physicists primarily as ways of interpreting these inferring techniques, and so of putting flesh on the mathematical skeleton." [438] p.34*

In the car engine example (section 3.1.6 on page 45) we saw how the different ways of modelling it effected not only our perception of its workings but also our method for interacting with it. These models depended crucially on the framework of mechanical understanding we had. If we had looked at a slightly different engine (e.g. a steam engine) we would form slightly different models of it, but these would very probably be on a par in terms of sophistication with those we formed of the car engine. The critical determinant of the complexity of our models arose not so much from the happenstance of the particular example as from our language of understanding and representation of engine mechanics.

I will, like others (e.g. Kauffman [249]), restrict my application of the concept of complexity to representations within specific language systems. Thus talking about complexity will necessitate indicating the language of representation that this is relative to. Here I intend the term "language" to have a wide interpretation which includes, but is not limited to, formal and natural languages. I am essentially using language as the most powerful idea available to capture the interaction and dependence of expressions of models to a framework.

35.See also Simon in [412].

The closeness of the relation between models and the language they are expressed in is implied in the following quote from Kuhn:

> *"So if anyone asks:' What more is there to look at in science besides the models, the actual phenomena, and the relationships between them?' we can answer 'The structure of the language used in a context where a scientific theory has been accepted.'" [161] p.44*