9 Appendix 2 - Longer Proofs

9.1 (Non-existence of) Complexity Measures on Strings

Definitions

Let L be the set of non-empty strings with symbols from $S = \{a, b, c, \ldots\}$, i.e. $L = \{a, aa, ab, b, \ldots\}$.

If x and y are the same pattern, but using different symbols, I will write $x \approx y$, e.g. $aab \approx bbc$.

Let $C : L \to \Re^+$ be a function to the non-negative reals (this is the intended complexity function.

I will write the concatenation of x and y as xy.

Let $\wp(x)$ be the set of non-empty substrings of x (including itself), e.g. $\wp(aba) = \{a, b, ab, ba, aba\}$.

I will show the substitution of y for $c \in S$ throughout x as $x(c/y)$, e.g. $aba(a/cd) = cdbcd$.

Two patterns are irrelevant to each other, $I(x, y)$, if they have no symbols in common, i.e. $I(x, y) \equiv \wp(x) \cap \wp(y) = \varnothing$, e.g. *I(aaba,cc)* but not *I(babcb,ddad)*.

When x and y are irrelevant to each other I will denote their concatenation: $x \cdot y$.

Let x^n be shorthand for the string x concatenated with itself *n-1* times, e.g. $(ab)^3 = ababab$.

In all of the below $w, x, y, z \in L$; $a, b, r, s, t \in S$; and $\alpha, i, n, m, s_1, s_2, \ldots, t_1, t_2, \ldots \in N$ (where N is the set of natural numbers).

Theorem

If S has at least three distinct symbols then there is no non-trivial measure from $\langle L, \cdot, \leq \rangle$ into $\langle \Re^+, +, \leq \rangle$, that is, if there *is* a function $C : L \to \Re^+$, defined on L to the non-negative real numbers such that, $\forall x, y \in L$:

$$C(x \cdot y) = C(x) + C(y) \qquad \text{(Irrelevant Join)}$$

$$x \leq y \Leftrightarrow C(x) \leq C(y) \qquad \text{(Subform)}$$

such that:

$$x \in \wp(y) \Rightarrow x \leq y \qquad \text{(Subpattern Property)}$$

$$I(x, y), s \in \wp(x) \Rightarrow C(x(s/y)) = C(x) + C(y) \quad \text{(Irrelevant Substitution)}$$

Then for all x, $C(x) = 0$.

There needs to be at least 3 symbols for the technical reason that sometimes you sometimes temporarily need a third symbol to apply "Irrelevant Substitution". For all of the below I use a constant, $k \geq 0$, defined as $C(ss)$. The triviality of the measure will come from a proof that $k = 0$.

Proof:

Lemma 1

$$C(s) = 0$$

Proof of Lemma:

Choose $t \in S$, $s \neq t$, now

$$C(t) = C(t(t/s)) = C(t) + C(s).$$

Lemma 2

If x has no repetitions of any symbols in it, then $C(x) = 0$.

Proof of Lemma:

Let $x = s_1 s_2 \ldots s_n$, where $s_i \in S$ are pairwise distinct.

Now using repeated instances of Irrelevant Join and Lemma 1 we have:

$$C(s_1 s_2 \ldots s_n) = C(s_1) + C(s_2 \ldots s_n) = \ldots = C(s_1) + \ldots + C(s_n) = 0.$$

Lemma 3:

If $x \approx y$, then $C(x) = C(y)$. (The *Substitution of Symbols*)

Proof of Lemma:

Let $s_1, s_2, \ldots, s_n$ be the symbols that occur in x and $t_1, t_2, \ldots, t_n$ be the corresponding symbols in *y*.

Now using Lemma 1 and repeated applications of Irrelevant Substitution we have:

$$C(x) = C(x) + C(t_1) + \ldots + C(t_n) = C(x(s_1/t_1) \ldots (s_n/t_n)) = C(y).$$

Lemma 4:

$$C\left(s^{n^m}\right) = mC\left(s^n\right).$$

Proof of Lemma:

By induction on *m*.

Base step: $C\left(s^{n^1}\right) = C\left(s^n\right) = 1 \cdot C\left(s^n\right)$.

Inductive step:

$$C\left(s^{n^{m+1}}\right) = C\left(s^{n \cdot n^m}\right) = C\left(t^{n^m}\left(t/s^n\right)\right) = C\left(t^{n^m}\right) + C\left(s^n\right) = C\left(s^{n^m}\right) + C\left(s^n\right)$$
$$= \left(mC\left(s^n\right) + C\left(s^n\right)\right) = (m+1)C\left(s^n\right).$$

Lemma 5:

$$C\left(s^{(2^m)}\right) = mk$$

Proof of Lemma:

This is a special case of Lemma 4, with *n=2*.

Lemma 6:

$$C\left(s^n\right) = k\log_2 n.$$

Proof of Lemma:

now given $m \geq 1$, choose $\alpha(m) > 0$ such that $2^\alpha \leq n^m \leq 2^{\alpha+1}$.
So a^{2^α} is a substring of a^{n^m} which is a substring of $a^{2^{\alpha+1}}$.

By the *subpattern* property, we have

$$C\left(a^{2^\alpha}\right) \leq C\left(a^{n^m}\right) \leq C\left(a^{2^{\alpha+1}}\right),$$

therefore by Lemmas 4 and 5: $k\alpha \leq mC\left(a^n\right) \leq k(\alpha+1)$,

so $k\dfrac{\alpha}{m} \le C\left(a^n\right) \le k\dfrac{\alpha+1}{m}$ since $m > 0$.

But by the choice of α

$\alpha \le m\log_2 n \le \alpha + 1$

so $k\dfrac{\alpha}{m} \le k\log_2 n \le k\dfrac{\alpha+1}{m}$, as $m > 0$.

Given that m can be arbitrarily large, $C\left(a^n\right)$ must be arbitrarily close to $k\log_2 n$.

Lemma 7:

$$C\left(x^n\right) = k\log_2 n + C(x).$$

Proof of Lemma:

Using Lemma 6, we have:

$$C\left(x^n\right) = C\left(s^n(s/x)\right) = C\left(s^n\right) + C(x) = k\log_2 n + C(x).$$

Lemma 8:

If x is a rotation of y then $C(x) = C(y)$

(where x is a rotation of y if $\exists w, z \in L\,(\,(wz = x) \wedge (zw = y)\,)$).

Proof of Lemma:

let w and z be strings such that $x = wz$, $y = zw$.

For $n > 1$, $(wz)^{n-1}$ is a sub-pattern of $(zw)^n$ which is a sub-pattern of $(wz)^{n+1}$, so by the sub-pattern property

$$C\left((wz)^{n-1}\right) \le C\left((zw)^n\right) \le C\left((wz)^{n+1}\right)$$

$$\Rightarrow k\log_2(n-1) + C(wz) \le k\log_2 n + C(zw) \le k\log_2(n+1) + C(wz)$$

(using lemma 7)

$$\Rightarrow k\log_2\left(\dfrac{n-1}{n}\right) + C(wz) \le C(zw) \le k\log_2\left(\dfrac{n+1}{n}\right) + C(wz)$$

Since n can be arbitrarily large, $C(wz) = C(zw)$.

Now to show that $k = 0$.

$$k + k\log_2 3$$

$= k + C\left(c^3\right)$	Lemma 6
$= k + C\left(c^3\right) + C(a) + C(b)$	Lemma 1
$= k + C\left(c^3\right) + C(ab)$	Irrelevant Join
$= k + C\left(c^3(c/(ab))\right)$	Irrelevant Substitution
$= k + C(ababab)$	Substitution expanded
$\geq k + C(ababa)$	Sub-pattern
$= k + C(abaab)$	Lemma 8
$= C\left(c^2\right) + C(abaab)$	Definition of k
$= C\left(c^2(c/(abaab))\right)$	Irrelevant Substitution
$= C(abaababaab)$	Substitution expanded
$\geq C(abaababaa)$	Sub-pattern
$= C(baaabaaba)$	Lemma 8
$\geq C(baaabaab)$	Sub-pattern
$= C(aabbaaab)$	Lemma 8
$\geq C(aabbaaa)$	Sub-pattern
$= C(aaaaabb)$	Lemma 8
$= C\left(a^5\right) + C\left(b^2\right)$	Irrelevant Join
$= k\log_2 5 + k.$	By the definition of k and Lemma 7

Finally I will show that $\forall x \in L; C(x) = 0$, which will prove the theorem.

Let x be a string containing the following n pairwise distinct symbols: $s_1, \ldots, s_n$.

Construct a new string $\hat{x} = f_n(f_{n-1}(\ldots f_1(x)\ldots))$ from x, using n applications of the following: $f_i(x)$ is constructed from $f_{i-1}(x)$ (using the convention that $f_0(x)$ is x) by replacing all instances of s_i in $f_{i-1}(x)$ that were also in x using these four steps:

(1) rotate x until the first symbol is the chosen instance of s_i;

(2) append the string $s_1 \ldots s_{i-1}$ to the front of the rotated string;

(3) rotate the new string until the chosen instance of s_i is at the end;

(4) append the string $s_{i+1} \ldots s_n$ at the end.

Fact 1:

$C(f_i(x)) \geq C(f_{i-1}(x))$ since a rotation does not change a string's complexity by Lemma 7 and appending symbols at the beginning or end of a string can only increase its complexity due to the subpattern assumption.

Fact 2:

By construction $\hat{x} = (s_1 \ldots s_n)^{|x|}$, since by construction every symbol in x is replaced by the string $s_1 \ldots s_n$.

By repeated applications of fact 1, we have $C(\hat{x}) \geq C(x)$.

So $C(x) \leq C(\hat{x}) \leq C\left((s_1 \ldots s_n)^{|x|} \right) = k \log_2 |x| + C(s_1 \ldots s_n) = 0$, since $k = 0$ and $s_1 \ldots s_n$ contains no symbol repetitions (Lemma 2).

....

9.2   Cyclomatic Number as a Lower Bound for Minimal Damage Cut

The cyclomatic number can be calculated as the number of arcs minus the number of nodes plus the number of disjoint partitions the graph is in. Each cut of the formula that incurs "damage" removes at least one arc and the formula is not fully "analysed" into components until you are left with a collection of trees and the cyclomatic number of a collection of trees is zero.

9.3   Decomposition of Formulas into Complexes

The proof deferred from section 5.3.2 on page 95.

Any formula can be decomposed into complexes. i.e.

For any $x \in \boldsymbol{L}$ there are $a_0, \ldots, a_n \in \boldsymbol{Cp}$ and $c_1, \ldots, c_n \in X_0$ such that

$$x = a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n$$

and for $i \leq n$, $\neg R(a_0{}^{c_1}/a_1 \ldots {}^{c_i}/a_i, c_i)$ so that

$$C(x) = C(a_0) + \ldots + C(c_n).$$

Note that this is not necessarily a unique decomposition.

<u>Proof:</u>

By induction on length of x:

<u>Base Step</u>

x is of length $1 \Rightarrow x \in X_0 \Rightarrow x \in \boldsymbol{Cp}$

and thus is its own decomposition.

<u>Induction Step</u>

$x \in \boldsymbol{Cp} \Rightarrow$ trivial decomposition of itself.

$x \notin \boldsymbol{Cp} \Rightarrow \exists y \in P(x) - \{x\} - X_0;\ \neg R(x^y/_c, y)$ Defn $\boldsymbol{Cp}$

Now by the induction hypothesis $x^y/_c$ and y have decompositions:

$x^y/_c \quad = a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n$

$y \quad\quad = b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m$

These two decompositions are irrelevant to each other, so

$x \quad\quad = (x^y/_c)^c/_y$

$\quad\quad\quad = a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n{}^c/b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m \quad$ as $\neg R(a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n, b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m)$

Now $\forall i < n\ \neg R(a_0{}^{c_1}/a_1 \ldots {}^{c_i}/a_i, a_{i+1}),\ \neg R(a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n, b_0)$

and for $\forall j < m\ \neg R(a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n{}^c/b_0{}^{k_1}/b_1 \ldots {}^{k_j}/b_j, b_{j+1}),$ since $\neg R(x^y/_c, y)$.

Now, of course,

$C(x) = C(a_0) + \ldots + C(a_n) +$

$\quad\quad\quad C(b_0) + \ldots + C(b_m).$

....

9.4   Generating a Measure from a Function on the Complexes

The proof from section 5.3.2 on page 95. First a lemma I will use in the proof:

Lemma

$y, z \in \wp(x),\ c, k \in X_0 - \wp(x),\ \neg R(x^y/_c, y),\ \neg R(x^z/_k, z)$

$\quad\quad \Rightarrow \neg R(y, z)$ or $y \in \wp(z)$ or $z \in \wp(y)$

<u>Proof:</u>

By induction on the depth of formulas.

<u>Base (depth 0):</u>

$depth(x)=0 \Rightarrow x \in X_0 \Rightarrow \wp(x)=\{x\}$

$y,z \in \wp(x) \Rightarrow y=x=z \Rightarrow y \in \wp(z)$

<u>Induction step:</u>

Assume lemma is true for all formulas of depth less than x ($depth(x)>0$).

One of:

(i) $x = uw$

(ii) $x = bst$

For some $w,s,t \in \mathbf{L}$, $u \in X_1$, $b \in X_2$, $depth(w)$, $depth(s)$, $depth(t) < depth(x)$

Case (i): $x = uw$

$y,z \in \wp(uw) = \{uw\} \cup \wp(w)$

There are essentially three possibilities: either (a) both y and z equal uw, (b) only one of them or (c) neither. Take each case in turn:

(a) $y=uw=z \Rightarrow y \in \wp(z)$

(b) (w.l.o.g.) $y=uw$, $z \in \wp(w) \Rightarrow z \in \wp(uw)=\wp(y)$

(c) $y,z \in \wp(w)$

by the inductive hypothesis since $\neg R(x^y/_c,y)$, $\neg R(x^z/_k,z) \Rightarrow \neg R(w^y/_c,y)$, $\neg R(w^z/_k,z)$ one of $\neg R(y,z)$, $y \in \wp(z)$ or $z \in \wp(y)$ holds.

Case (ii): $x = bst$

$y,z \in \wp(bst) = \{bst\} \cup \wp(s) \cup \wp(t)$

Here there are essentially four possibilities (modulus the obvious symmetries of s and t): either (a) both y and z equal bst, (b) one is equal to bst and the other is a member

of $\wp(s)$, (c) both are members of $\wp(s)$ or (d) one is a member of $\wp(s)$ and the other a member of $\wp(t)$. Taking each possibility in turn:

(a) $y=bst=z \Rightarrow y\in \wp(z)$

(b) (w.l.o.g.) $y=bst$, $z\in \wp(s) \Rightarrow z\in \wp(bst)=\wp(y)$

(c) (w.l.o.g.) $y,z\in \wp(s)$

$\neg R(x^y/_c,y)$, $\neg R(x^z/_k,z) \Rightarrow \neg R(s^y/_c,y)$, $\neg R(s^z/_k,z)$

and so by the inductive hypothesis one of $\neg R(y,z)$, $y\in \wp(z)$ or $z\in \wp(y)$ holds.

(d) (w.l.o.g.) $y\in \wp(s)-\wp(t)$, $z\in \wp(t)-\wp(s)$

$\neg R(x^y/_c,y)$

$\quad \Rightarrow \neg R((bst)^y/_c,y)$ $\qquad\qquad\qquad\qquad$ as $x=bst$

$\quad \Rightarrow \neg R(b(s^y/_c)t,y)$ $\qquad\qquad\qquad\qquad$ $y\notin \wp(t)$

$\quad \Rightarrow \wp(b(s^y/_c)t)\cap \wp(y)=\varnothing$ $\qquad\qquad$ Defn R

$\quad \Rightarrow [\{b(s^y/_c)t\}\cup \wp(s^y/_c)\cup \wp(t)]\cap \wp(y)=\varnothing$

$\quad \Rightarrow \wp(t)\cap \wp(y)=\varnothing$

Similarly

$\neg R(x^z/_k,y) \Rightarrow \wp(s)\cap \wp(z)=\varnothing$

Now if $q\in \wp(x)$ then either: $q=bst$, $q\in \wp(s)$ or $q\in \wp(t)$. If $q=bst$ then $q\notin \wp(y)\cap \wp(z)$ by the assumptions of (d). If $q\in \wp(s)$ then $q\notin \wp(z)$ since $\wp(s)\cap \wp(z)=\varnothing$. If $q\in \wp(t)$ then $q\notin \wp(y)$ since $\wp(t)\cap \wp(y)=\varnothing$. In any of these cases $q\notin \wp(y)\cap \wp(z)$ so we have $\wp(y)\cap \wp(z)=\varnothing$, i.e. $\neg R(y,z)$.

This completes the proof of possibility (d), hence cases *(i)* and *(ii)* and thus the induction step.

....

Now for the main proof.

A function, g, on **Cp** in **L**, g:**Cp**→$\Re^+$,

$x \in X_0 \Rightarrow g(x)=0$

$x \approx y \Rightarrow g(x)=g(y)$

$x \leq y \Rightarrow g(x) \leq g(y)$.

will generate a unique complexity measure, C:**L**→$\Re^+$, on **L**, thus:

$C(x) \equiv_{\text{dff}} g(a_0) + g(a_1) + g(a_2) + \ldots + g(a_n)$,

for some decomposition: $x = a_0{}^{c_1}/a_1{}^{c_2}/a_2 \ldots {}^{c_n}/a_n$, into complexes: $a_0, a_1, a_2, \ldots, a_n \in$ **Cp**, as above and hence an ordering, $\leq$, on **L**:

$x \leq y \Leftrightarrow C(x) \leq C(y)$.

Proof:

We have to check two things:

That C is well-defined, that is given two decompositions of x the sum of g over these is the same, that is:

(1) $x = a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n = b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m$

$\Rightarrow g(a_0) + g(a_1) + \ldots + g(a_n) = g(b_0) + g(b_1) + \ldots + g(b_n)$.

And that the complexity of the result of an irrelevant substitution is the sum of the complexity of the formula that is substituted in and the formula it is substituted into:

(2) $k \in X_0 \cap \wp(x)$, $\neg R(x,y)$

$\Rightarrow C(x^k/_y) = C(x) + C(y)$

I do this by induction on the maximum size of formulas.

Base (size 1)

$|x|=1 \Rightarrow x \in X_0 \Rightarrow C(x) = 0 = g(x)$

and so C is well defined, showing (1).

$k \in X_0 \cap \wp(x)$

$\Rightarrow k=x$

$\Rightarrow x^k/_y = y$

$\Rightarrow C(x^k/_y) = C(y) = C(y) + 0 = C(y) + g(x) = C(y) + C(x)$

showing (2).

Induction Step

Assume (1) and (2) hold for all formulas with size less than that of x.

Suppose that x can be decomposed by two different non-trivial irrelevant substitutions. So that:

$$\exists y,z \in \wp(x)-\{x\}, \exists c,k \in X_0-\wp(x), \neg R(x^y/_c,y), \neg R(x^z/_k,z)$$

If we could show that $C(x^y/_c) + C(y) = C(x^z/_k) + C(z)$, then by the induction hypothesis C would be a well defined measure on $x^y/_c$, y, $x^z/_k$ and z and thus we would have shown (1) for all formulas up to and including size $|x|$.

Now if $k \in \wp(x) \cap X_0$, $\neg R(x,y)$, x and y have decompositions into complexes:

$$x = a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n, \forall i \leq n \; \neg R(a_0{}^{c_1}/a_1{}^{c_2}/a_2 \ldots {}^{c_i}/a_i, a_{i+1})$$

$$y = b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m, \forall j \leq m \; \neg R(b_0{}^{k_1}/b_1 \ldots {}^{k_j}/b_j, b_{j+1})$$

$\Rightarrow C(x^k/_y)$

$$= C(a_0{}^{c_1}/a_1 \ldots {}^{c_n}/a_n(k/(b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m)))$$

$$= C(a_0{}^{c_1}/a_1 \ldots {}^{c_i}/(a_i({}^k/(b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m))) \ldots {}^{c_n}/a_n) \qquad \text{as for one } i \leq n, k \in a_i$$

$$= C(a_0{}^{c_1}/a_1 \ldots {}^{c_i}/a_i{}^k/(b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m) \ldots {}^{c_n}/a_n) \qquad \text{as } k \text{ only occurs in } a_i$$

$$= C(a_0{}^{c_1}/a_1 \ldots {}^{c_i}/a_i{}^k/b_0{}^{k_1}/b_1 \ldots {}^{k_m}/b_m \ldots {}^{c_n}/a_n) \text{as } k_1, \ldots k_m \text{ do not occur in } a_0, \ldots a_i$$

$$= g(a_0) + \ldots + g(a_i) + g(b_0) + \ldots + g(b_m) + g(a_{i+1}) + \ldots + g(a_n)$$

since C is well-defined on $a_0, \ldots a_n, b_0, \ldots b_m$ by induction hypothesis

$$= g(a_0) + \ldots + g(a_n) + g(b_0) + \ldots + g(b_m) \qquad \text{by rearranging}$$

$$= C(x) + C(y)$$

So all we have left to prove is $\exists y,z \in \wp(x)-\{x\}, \exists c,k \in X_0-\wp(x), \neg R(x^y/_c,y), \neg R(x^z/_k,z)$

$\Rightarrow C(x^y/_c) + C(y) = C(x^z/_k) + C(z)$ for formula of size $|x|$.

Using the Lemma immediately above, either:

(i) $\neg R(x,y)$

(ii) $y \in \wp(z)$ or $z \in \wp(y)$.

<u>Case (i)</u>

$\neg R(x^y/_c, y) \Rightarrow \neg R(x^z/_k{}^y/_c, y)$ as $\neg R(k, y)$

   $\Rightarrow C(x^z/_k) = C(x^z/_k{}^y/_c) + C(y)$ by induction hypothesis as $|x^z/_k| < |x|$

similarly $C(x^y/_c) = C(x^y/_c{}^z/_k) + C(z)$

Now x and y occur separately in x so $x^z/_k{}^y/_c = x^y/_c{}^z/_k$, and thus:

$C(x^y/_c) - C(z) = C(x^y/_c{}^z/_k) = C(x^z/_k{}^y/_c) = C(x^z/_k) - C(y)$

<u>Case (ii)</u>

 $y \in \wp(z)$ or $z \in \wp(y)$.

W.l.o.g. say, $y \in \wp(z)$.

 $\neg R(x^z/_k, z) \Rightarrow \neg R(x^y/_c{}^{(zy/c)}/_k, (z^y/_c))$ as $y \in \wp(z)$, $y \notin \wp(x^z/_k)$

 $C(x^y/_c) = C(x^y/_c{}^{(zy/c)}/_k{}^y/(z^y/_c))$ as $x^y/_c = x^y/_c{}^{(zy/c)}/_k{}^y/(z^y/_c)$

    $= C(x^y/_c{}^{(zy/c)}/_k) + C(z^y/_c)$ as $\neg R(x^y/_c{}^{(zy/c)}/_k, (z^y/_c))$

    $= C(x^z/_k) + C(z^y/_c)$ as $x^y/_c{}^{(zy/c)}/_k = x^z/_k$

 $\neg R(x^y/_c, y), z^y/_c \in \wp(x^y/_c) \Rightarrow \neg R(z^y/_c, y)$

 $C(z) = C(z^y/_c{}^c/_y) = C(z^y/_c) + C(y)$ as $\neg R(z^y/_c, y)$

 $C(x^y/_c) = C(x^z/_k) + C(z) - C(y)$ substituting out $C(z^y/_c)$

Which rearranged shows: $C(x^y/_c) + C(y) = C(x^z/_k) + C(z)$

....

9.5   Three Conditions that are Equivalent on a Weak Complexity Measure

This is the proof from section 5.3.5 on page 104. But first I show that condition (i) implies the subformula property.

(i) $\Rightarrow$ the Subformula Property

If $R(x,y) \Rightarrow C(bxy) > \max\{C(x), C(y)\}$ then

   $y \in P(x) \Rightarrow C(y) \leq C(x)$

Proof:

The case $y=x$ is trivial so assume $y \neq x$.

By induction on depth of x:

Base:

$x \in X_0 \Rightarrow y \in X_0 \Rightarrow C(y)=0=C(x)$

Induction Step:

x is of the form

   (1) uz

or (2) bzw

Case (a): $x=uz$

   $y \in P(z)$ as $y \neq x$

   $\Rightarrow C(y)$

	$\leq C(z)$	by induction hypothesis
	$= C(uz) = C(x)$	

Case (b): $x=bzw$

   $\Rightarrow y \in \wp(z)$ or $y \in \wp(w)$, say w.l.o.g. $y \in \wp(z)$

   if $\neg R(z,w) C(x) = C(bzw)$

	$= C(z) + C(w)$	by Irrel Join
	$\geq C(y) + C(w)$	by induction hypothesis
	$\geq C(y)$	

   if $R(z,w)$ $C(x) = C(bzw)$

	$> \max\{C(y), C(w)\}$	by Rel Join
	$\geq C(y).$	

   ....

The following three conditions are equivalent on a weak complexity measure, for $b \in X_2$, $x, y \in L$:

(i) $R(x,y) \Rightarrow C(bxy) > \max\{C(x), C(y)\}$ (Rel Join)

(ii) $y \in \wp(x) - \{x\}$, $c \in X_0 - (x)$, $R(x^y/_c, y) \Rightarrow C(y) < C(x)$ (Rel Subform)

(iii) $y \in \wp(x) - \{x\} - X_0$, $x \in \mathbf{Cp} \Rightarrow C(y) < C(x)$ (Subform of Complex)

Note that I have written them here in terms of the ordering on the reals rather than the ordering $\leq$ on L, for convenience. The equivalence is assured due to the homomorphic mapping between them.

Proof: (i) $\Rightarrow$ (ii)

By induction on the maximum depth of x.

Assuming $y \in \wp(x) - \{x\}$, $c \in \wp X_0 - (x)$, $R(x^y/_c, y)$

Base:

$x \in X_0$, vacuously true as $\wp(x) - \{x\} = \varnothing$

Induction step:

either *(a)* x is form (uz) for some $u \in X_1$, $z \in L$

or *(b)* x is of form (bwz) for some $b \in X_2$, $w, z \in L$

Case (a): $x=uz$

$c \in X_0 - \wp(x) \Rightarrow c \in X_0 - \wp(y)$

$\neg R(uc,y) \Rightarrow \neg R(uy^y/_c,y)$

$\qquad \Rightarrow \neg R(x^y/_c,y)$

$\qquad \Rightarrow y \neq uy$ $\hspace{6cm}$ otherwise $R(uy^y/_c,y)$

$\qquad \Rightarrow y \neq z$

$\qquad y \in \wp(z)$ $\hspace{6cm}$ as $y \in \wp(uz) = \wp(x)$

$R(x^y/_c,y) \Rightarrow R(uz^y/_c,y)$

$\qquad \Rightarrow R(z^y/_c,y)$

$\Rightarrow C(y)$

$\qquad < C(z)$ $\hspace{6cm}$ by the induction hypothesis

$\qquad = C(uz)$

$\qquad = C(x).$

Case (b): $x=bwz$

If $R(w,z)$

$x \neq y \Rightarrow y \in \wp(w)$ or $y \in \wp(z)$, say w.l.o.g. $y \in \wp(w)$

$C(x) \quad = C(bwz)$

$\qquad > \max\{C(w),C(z)\}$ $\hspace{4.5cm}$ from (i).

$\qquad \geq \max\{C(y),C(z)\}$ $\hspace{3cm}$ as $y \in P(w) \Rightarrow C(y) \leq C(w)$

$\qquad \geq C(y)$

If $\neg R(w,z)$

$y \neq w$ and $y \neq z$

as, if not, say (w.l.o.g.) y=w then:

$R(x^y/_c,y) \Rightarrow R(bwz^y/_c,y)$ by assumption Case (b)

$\Rightarrow R(byz^y/_c,y)$ by assumption y=w

$\Rightarrow R(bc(z^y/_c),y)$

$\Rightarrow R(z^y/_c,y)$ as $\neg R(y,c)$ by choice of c

$\Rightarrow R(z,y)$

$\Rightarrow R(z,w)$ as y=w

which is contrary to assumption.

$\Rightarrow y \in \wp(z)-\{z\}$ or $y \in \wp(w)-\{w\}$

$R(x^y/_c,y) \Rightarrow R(z^y/_c,y)$ or $R(w^y/_c,y)$,   as $\neg R(w,z)$

say w.l.o.g. $R(z^y/_c,y)$ $[\Rightarrow \neg R(w^y/_c,y)$ as $\neg R(w,z)]$

If $y \in (z)$, $y \neq z$ then

$C(y) < C(z)$ by induction hypothesis

$\leq C(x)$ as $z \in \wp(x)$

If $y \in \wp(w)-\{w\}$

then $R(w^y/_c,y)$

or else if $\neg R(w^y/_c,y)$:

$R(x^y/_c,y) \Rightarrow R((bwz)^y/_c,y)$

$\Rightarrow R(b(w^y/_c)(z^y/_c),y)$

$\Rightarrow R(z^y/_c,y)$ as $\neg R(w^y/_c,y)$

$\Rightarrow R(z,y)$

$\Rightarrow R(z,w)$ as $y \in \wp(w)$

which is contrary to the *Case (b)* assumption

$\Rightarrow C(y) < C(w)$ by the induction hypothesis

$\leq C(x)$ as $w \in \wp(x)$

....

Proof: (ii) $\Rightarrow$ (i)

$R(x,y) \Rightarrow R(bxc,y)$ for suitably chosen $c \in X_0$

      $\Rightarrow R((bxy)^y/_c, y)$

Now $y \in \wp(bxy) - \{bxy\}$ so $C(y) < C(bxy)$ by (ii)

Likewise $C(x) < C(bxy)$

      $\Rightarrow C(bxy) > \max\{C(y), C(x)\}.$

....

Proof: (iii) $\Rightarrow$ (ii)

$y \in \wp(x) - \{x\}$, $c \in X_0 - \wp(x)$, $R(x^y/_c, y)$

By induction on the length of the decomposition of x.

Base:

$x \in \boldsymbol{Cp} \Rightarrow C(y) < C(x)$ by (iii)

Induction Step:

$x \notin \boldsymbol{Cp} \Rightarrow$ there is a $z \in \wp(x) - \{x\} - X_0$, $z \in \boldsymbol{Cp}$; $\neg R(x^z/_k, z)$ $k \in X_0 - \wp(x)$

      $\Rightarrow y \neq z$ as $R(x^y/_c, y)$

      $\Rightarrow C(x) = C(x^z/_k) + C(z)$ by Irrel Subs

(*) $\Rightarrow C(x^z/_k) \leq C(x)$ and $C(z) \leq C(x)$

$y \in \wp(x)$, $\neg R(x^z/_k, z)$ means either

      (i) $y \in \wp(x^z/_k)$

or   *(ii)* $y \in \wp(z)$

Case (i):

if $y = x^z/_k$ then $x^z/_k \in \wp(x)$ but $\neg R(k, x)$ as $k \in X_0 - \wp(x)$

so $y \neq x^z/_k$.

$R(x^y/_c, y) \Rightarrow R(x^z/_k{}^y/_c, y)$ as $\neg R(x^z/_k, z)$ and $y \in \wp(x^z/_k)$

      $\Rightarrow C(y) < C(x^z/_k)$ by induction hypothesis

      $\leq C(x)$ from (*)

Case (ii):

z∈ **Cp**

$\Rightarrow C(y) < C(z)$ by (iii)

$\leq C(x)$ from (*).

....

Proof: (ii) $\Rightarrow$ (iii)

$y \in \wp(x)-\{x\}$, $x \in$ **Cp**

$x \notin X_0$ as $\wp(x)-\{x\} \neq \varnothing$

$y \notin X_0$

$\Rightarrow R(x^y/_c, y)$ by Defn **Cp**

$\Rightarrow C(y) < C(x)$ by (ii).

....